

ON CONVEXITY OF THE REGULAR SET OF CONICAL KÄHLER-EINSTEIN METRICS

VED V. DATAR*

ABSTRACT. In this note we prove convexity, in the sense of Colding-Naber, of the regular set of solutions to some complex Monge-Ampère equations with conical singularities along simple normal crossing divisors. In particular, any two points in the regular set can be joined by a smooth minimal geodesic lying entirely in the regular set. We show that as a result, the classical theorems of Myers and Bishop-Gromov extend almost verbatim to this singular setting.

1. Introduction

Let $(X, \hat{\omega})$ be a Kähler manifold with a smooth reference Kähler metric $\hat{\omega}$. A divisor

$$(1.1) \quad D = \sum_{j=1}^N (1 - \beta_j) D_j$$

with D_j a smooth irreducible divisor, and $\beta_j \in (0, 1)$ is called a *simple normal crossing* divisor if locally at any $p \in D$ lying in the intersection of exactly k divisors D_1, \dots, D_k , there exists a coordinate chart $(U, (z_1, \dots, z_n))$ such that $D_j|_U$ is cut out by $[z_j = 0]$ for $j = 1, \dots, k$. A conical Kähler metric along D , is then a smooth Kähler metric on $X \setminus D$ such that the restriction to U is equivalent to the following model edge metric

$$(1.2) \quad \omega_e = \sum_{j=1}^k |z_j|^{-2(1-\beta_j)} dz_j \wedge d\bar{z}_j + \sum_{j=k+1}^N dz_j \wedge d\bar{z}_j$$

In this note, we are concerned with Kähler currents $\omega = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi$, with $\varphi \in L^\infty(X) \cap PSH(X, \hat{\omega})$, and solving the following singular complex Monge-Ampère equation :

$$(1.3) \quad \begin{cases} (\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \frac{e^{-\lambda \varphi} \Omega}{\prod_{j=1}^N |s_j|_{h_j}^{2(1-\beta_j)}} \\ \omega = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \end{cases}$$

where s_j is the defining section of D_j , h_j is a smooth hermitian metric on the line bundle generated by D_j , and Ω is a smooth volume form satisfying

$$(1.4) \quad \sqrt{-1} \partial \bar{\partial} \log \Omega + \lambda \hat{\omega} + \chi = \sum_{j=1}^N (1 - \beta_j) \sqrt{-1} \partial \bar{\partial} \log h_j$$

for some smooth $(1, 1)$ form χ . The Ricci curvature of ω solves a twisted conical Kähler-Einstein equation

$$(1.5) \quad Ric(\omega) = \lambda \omega + \chi + [D]$$

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where $[D]$ is the current of integration along D .

Kähler metrics with cone singularities have played an important role in the recent breakthrough on the existence of Kähler-Einstein metrics on Fano manifolds [6, 29]. A detailed study of such equations was first carried out by Yau [31] in his seminal paper on the Calabi conjecture. Among other things, he proved that any bounded solution to (1.3) is smooth away from the divisor D . More recently, when the divisor is $(1 - \beta)D$ i.e has only some smooth component, very precise regularity results for solutions to the equation were obtained by Brendle [1] in the case when $\beta < 1/2$, and for all $\beta \in (0, 1)$ by Donaldson [13], Jeffres-Mazzeo-Rubinstein [19], and Chen-Donaldson-Sun [8].

Unfortunately, many linear systems do not contain smooth divisors. So, for geometric applications, it is important to address the questions of regularity for cone angles along normal crossing divisors. The first step in this direction are the results of Campana-Guenancia-Păun [2], and Guenancia-Păun [17], that any ω solving (1.3) is locally equivalent to the standard edge metric (1.2) (cf. [15] for a shorter proof of their results). While higher regularity results are awaited, the aim of this note is to prove the convexity of $X \setminus D$, in the sense of Colding-Naber, with respect to the metric induced by ω . This allows the extension of the classical comparison theorems to the conical setting, and might be useful in studying the moduli space of Kähler-Einstein metrics with cone singularities (cf. [14]).

Since ω is smooth on $X \setminus D$, it defines a length functional \mathcal{L}_ω and in turn, a distance function

$$d_\omega(p, q) = \inf \{ \mathcal{L}_\omega(\gamma) \mid \gamma : [0, 1] \rightarrow X \setminus D \text{ piecewise smooth}, \gamma(0) = p, \gamma(1) = q \}$$

Since, ω is locally equivalent to a model edge metric [2, 17, 15], it is easily seen that the metric completion of $X \setminus D$ under this distance function is homeomorphic to X itself, and we set

$$(X, d) = \overline{(X \setminus D, d_\omega)}$$

where the bar denotes the metric completion. We first prove an approximation theorem for ω , extending results of [7, 29] for conical Kähler-Einstein metrics in the Fano case.

Proposition 1.1. *Let $\omega = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi$ be a solution to (1.3) with $\varphi \in PSH(X, \hat{\omega}) \cap L^\infty(X)$. Then there exist uniform constants $A, \Lambda \gg 1$, and a sequence $\omega_\eta \in [\omega]$ of smooth Kähler metrics such that*

$$(1) \text{ Ric}(\omega_\eta) > -A\omega_\eta ; \text{ diam}(X, \omega_\eta) < \Lambda$$

$$(2) \text{ As } \eta \rightarrow 0,$$

$$(X, \omega_\eta) \xrightarrow{d_{GH}} (X, d)$$

where (X, d) as above, is the metric completion of $(X \setminus D, d_\omega)$.

It should be noted that in the Fano case with $\chi = 0$ and a smooth pluri-canonical divisor D , Chen-Donaldson-Sun [7] and Tian [29], prove a much stronger result, namely one can approximate with the *same* Ricci lower bound as the conical metric. For such a result, it is of course necessary that X is Fano.

Next, recall that a unit-speed path $\gamma : [0, l] \rightarrow X$ joining p, q is said to be a *minimal geodesic* if $d(p, q) = l$. It is said to be a *limiting geodesic* if there exists a sub-sequence $\{\eta_j\}$ with unit-speed ω_{η_j} -geodesics $\gamma^{\eta_j} : [0, l_j] \rightarrow X$ such that $l_j \rightarrow l$ and $\gamma^{\eta_j} \rightarrow \gamma$ point wise. Limiting geodesics can usually be found in abundance. Our main theorem is :

Theorem 1.1. $X \setminus D \subset (X, d)$ is geodesically convex, in the following sense: if any interior point of a limiting minimal geodesic lies in $X \setminus D$, then all the interior points must lie in $X \setminus D$.

The theorem is proved by combining the above smoothening with the results of Colding-Naber [11] on the Hölder continuity of tangent cones for limit spaces. It must be noted that the theorem does not rule out the possibility of *some* geodesic connecting $p, q \in X \setminus D$ passing through D , though it is expected that such a scenario will not occur. A nice consequence of the above theorem is the following

Corollary 1.1. Let $p, q \in X \setminus D$ with $l = d(p, q)$. Then there exists a smooth unit speed geodesic $\gamma : [0, l] \rightarrow X \setminus D$ with $\gamma(0) = p$ and $\gamma(l) = q$.

Notation. Distances with respect to ω_η and ω are denoted by d_η, d respectively. Paths connecting points p, q are denoted by γ_{pq} . Minimal geodesics are denoted with superscripts to specify the reference metric. For example, d_η -minimal and d -minimal geodesics are denoted by γ_{pq}^η and γ_{pq}^d respectively. The lengths of paths are denoted by \mathcal{L}_ω and \mathcal{L}_η respectively.

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2. Approximation and smooth convergence away from D .

We first smoothen out the Kähler current ω , following the same strategy as in [2, 7, 29]. By Demailly's regularization theorem [12], there exists a sequence $\psi_\eta \in C^\infty(X) \cap PSH(X, \hat{\omega})$ such that $\psi_\eta \searrow \varphi$ point wise as $\eta \rightarrow 0$. Note that all the ψ_η 's are uniformly bounded in the L^∞ norm. The metrics ω_η are then constructed as the solutions to the following perturbation of (1.3)

$$(2.6) \quad \begin{cases} (\hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_\eta)^n = \frac{e^{-\lambda\psi_\eta + c_\eta\Omega}}{\prod_{j=1}^N (|s_j|_{h_j}^2 + \eta)^{(1-\beta_j)}} \\ \omega_\eta = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_\eta > 0 \end{cases}$$

where c_η is a constant such that the integrals on both sides are equal. By Yau's work on the Calabi conjecture [31], there always exists a smooth solution to the above equation for $\eta > 0$. It is easy to see that $|c_\eta|$ is uniformly bounded, and in fact tends to zero as $\eta \rightarrow 0$.

Lemma 2.1. If $\omega_\eta = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_\eta$ is a solution to (2.6), then there exists an $A \gg 1$ such that

$$Ric(\omega_\eta) > -A\hat{\omega}$$

Proof. We follow the computation in [7]. First observe that for any smooth $f > 0$

$$\sqrt{-1}\partial\bar{\partial}\log(f + \eta) \geq \frac{f}{f + \eta} \sqrt{-1}\partial\bar{\partial}\log f$$

So, on $X \setminus D$

$$\begin{aligned}
Ric(\omega_\eta) &= \lambda \sqrt{-1} \partial \bar{\partial} \psi_\eta - \sqrt{-1} \partial \bar{\partial} \log \Omega + \sum_{j=1}^N (1 - \beta_j) \sqrt{-1} \partial \bar{\partial} \log (|s_j|_{h_j}^2 + \eta) \\
&\geq -\lambda \hat{\omega} - \sqrt{-1} \partial \bar{\partial} \log \Omega + \sum_{j=1}^N (1 - \beta_j) \frac{|s_j|_{h_j}^2}{(|s_j|_{h_j}^2 + \eta)} \sqrt{-1} \partial \bar{\partial} \log |s_j|_{h_j}^2 \\
&= -\sum_{j=1}^N (1 - \beta_j) \frac{\eta}{(|s_j|_{h_j}^2 + \eta)} \sqrt{-1} \partial \bar{\partial} \log h_j + \chi \quad (\text{by equation 1.4}) \\
&\geq -A \hat{\omega}
\end{aligned}$$

if for instance $-\sqrt{-1} \partial \bar{\partial} \log h_j > -A \hat{\omega}/2N$, and $\chi > -A \hat{\omega}/2$. We also use the fact that $1 - \beta_j \geq 0 \forall j$ in the second line. So this argument will not work if the divisor is not effective. \square

Next, we obtain uniform C^0 and C^2 estimates on φ_η .

Proposition 2.1. *There exists a constant $C = C(n, A, \|\omega^n/\Omega\|_{L^{1+\delta}(X, \Omega)}, \|Rm(\hat{\omega})\|) \gg 1$ independent of η , such that*

(1)

$$\|\varphi_\eta\|_{C^0(X)} < C$$

(2)

$$C^{-1} \hat{\omega} < \omega_\eta < \frac{C \hat{\omega}}{\prod_{j=1}^N (|s_j|_{h_j}^2 + \eta)^{(1-\beta_j)}}$$

Proof. The proof is standard. The right hand side of equation (2.6) is uniformly in $L^{1+\varepsilon}(X, \omega)$ for some $\varepsilon > 0$, since all the β_j 's are strictly positive, $\psi_\eta, |c_\eta|$ are uniformly bounded, and D is a simple normal crossing divisor. The C^0 estimate now follows directly from the work of Kolodziej [20, 21]. For the C^2 estimate, we consider the following quantity :

$$(2.7) \quad Q = \log \text{tr}_{\omega_\eta} \hat{\omega} - B \varphi_\eta$$

By Lemma 2.1, $Ric(\omega_\eta) > -A \hat{\omega}$ for some $A \gg 1$. Then by the Chern-Lu inequality, there exist constants $B, C \gg 1$ depending on A , the dimension n , and an upper bound for the bisectional curvature of $\hat{\omega}$, such that

$$\Delta_\eta Q \geq \text{tr}_{\omega_\eta} \hat{\omega} - C$$

By maximum principle and the uniform C^0 estimate,

$$\text{tr}_{\omega_\eta} \hat{\omega} \leq C$$

But then using the equation (2.6), and an elementary arithmetic-geometric mean inequality

$$\begin{aligned}
\text{tr}_{\hat{\omega}} \omega_\eta &\leq (\text{tr}_{\omega_\eta} \hat{\omega})^{n-1} \frac{\omega_\eta^n}{\hat{\omega}^n} \\
&\leq \frac{C}{\prod_{j=1}^N (|s_j|_{h_j}^2 + \eta)^{(1-\beta_j)}}
\end{aligned}$$

\square

Higher order estimates away from the divisor follow by standard methods (cf. [23]). As a straightforward corollary we have,

Corollary 2.1. *If the ω_η are solutions to (2.6) then there exist constants $A, \Lambda > 0$ such that*

- (1) $Ric(\omega_\eta) > -A\omega_\eta$; $diam(X, \omega_\eta) < \Lambda$.
- (2) *Locally on $X \setminus D$, as $\eta \rightarrow 0$,*

$$\omega_\eta \xrightarrow{C_{loc}^\infty(X \setminus D)} \omega$$

- (3) *For all open sets $U \subset X$,*

$$Vol_{\omega_\eta}(U) \xrightarrow{\eta \rightarrow 0} Vol_\omega(U)$$

3. Almost geodesic convexity and proof of Gromov-Hausdorff convergence

The proof of the Gromov-Hausdorff convergence follows along the lines of [32, 29, 14]. The main technical ingredient is the following relative comparison lemma of Gromov [16].

Lemma 3.1. *Let (M, g) , be a Riemannian manifold of dimension m and $T \subset M$ be any compact set with a smooth boundary, such that*

•

$$Ric(g) > -Ag \text{ ; } diam(M, g) < \Lambda$$

- *For some points $p_1, p_2 \in M$ with $B(p_j, \varepsilon) \cap T = \emptyset$ for $j = 1, 2$, every minimal geodesic from p_1 to points in $B(p_2, \varepsilon)$ intersects T .*

Then, there exists a constant $c = c(n, \varepsilon, A, \Lambda)$ such that

$$Vol(\partial T, g) > cVol(B(p_2, \varepsilon), g)$$

The next lemma proves that for almost all points, $X \setminus D$ is geodesic convex. This was proved by Cheeger-Colding [5] for Gromov-Hausdorff limits of Riemannian manifolds with a Ricci lower bound. In our case, we haven't yet identified the Gromov-Hausdorff limit with X , and so we give an elementary proof using the above comparison lemma and smooth convergence on $X \setminus D$.

Lemma 3.2. *Let $K \subset\subset X \setminus D$, and $d(\partial K, D) > 4\varepsilon$. Then there exists a $\delta = \delta(n, \varepsilon, A, \Lambda)$, such that if T is a neighborhood of D in $X \setminus K$ with $d(p, \partial T) > 2\varepsilon \forall p \in K$, and $Vol_d(\partial T) < \delta$, then, for all $p, q \in K$, there exists a $q' \in B_d(q, \varepsilon)$ and a minimal d -geodesic $\gamma_{pq'}^d : [0, l] \rightarrow X \setminus T$ connecting p to q' .*

*Proof. **Claim 1:** If η is sufficiently small,*

$$B_{d_\eta}(q, \varepsilon/2) \subset B_d(q, \varepsilon)$$

Suppose not, then for arbitrarily small η , there exists an $x \in X$ such that $d_\eta(q, x) < \varepsilon/2$, but $d(q, x) > \varepsilon$. The minimal η -geodesic γ_{qx}^η has a first point of contact $\tilde{x} \in \partial B_d(q, \varepsilon)$. Then $\mathcal{L}_\omega(\gamma_{q\tilde{x}}^\eta) \geq \varepsilon$, and hence $d_\eta(q, \tilde{x}) = \mathcal{L}_\eta(\gamma_{q\tilde{x}}^\eta) > 3\varepsilon/4$ if η is sufficiently small, by uniform smooth convergence on $X \setminus T$ and the fact that γ_{qx}^η is minimal. This is a contradiction and the claim is proved.

Claim-2: For η small, and any $p, q \in K$, there exists $q_\eta \in B_{d_\eta}(q, \varepsilon/2)$ and a minimal unit speed ω_η -geodesic $\gamma_{pq_\eta}^\eta : [0, l_\eta] \rightarrow X \setminus T$.

If not, then by volume comparison, diameter bound, Lemma 3.1, and volume convergence (cf. Lemma 2.1),

$$c\kappa\varepsilon^{2n} \leq cVol_{\omega_\eta}(B_{d_\eta}(q, \varepsilon/2)) \leq Vol_{\omega_\eta}(\partial T) \leq 2Vol_\omega(\partial T) \leq 2\delta$$

Pick $\delta = c\kappa\varepsilon^{2n}/4$ to get a contradiction.

So there is a sequence of points $q_\eta \in B_{d_\eta}(q, \varepsilon/2) \subset B_d(q, \varepsilon)$ and η -minimal geodesics $\gamma_{pq_\eta}^\eta \subset X \setminus T$. Since the convergence is smooth on $X \setminus T$ and the diameter is uniformly bounded, by Ascoli-Arzelà there exists a $q' \in B_d(q, \varepsilon)$ and a limiting geodesic $\gamma_{pq'} : [0, l] \rightarrow X \setminus T$ from p to q' .

Claim-3: $\gamma_{pq'}$ is d -minimal. i.e

$$\mathcal{L}_\omega(\gamma_{pq'}) = d(p, q')$$

If not, then by definition of d , there exists a path $\tilde{\gamma}_{pq'} : [0, 1] \rightarrow X \setminus D$ such that $\mathcal{L}_\omega(\tilde{\gamma}_{pq'}) < \mathcal{L}_\omega(\gamma_{pq'}) - \zeta$, for some $\zeta > 0$. For η small, $d(q', q_\eta) < \zeta/8$. The minimal d -geodesic $\gamma_{q'q_\eta}^d$ doesn't hit ∂T . So once again by smooth convergence $\mathcal{L}_\eta(\gamma_{q'q_\eta}^d) < \zeta/4$. On the other hand, for η small,

$$\mathcal{L}_\eta(\tilde{\gamma}_{pq'}) < \mathcal{L}_\omega(\tilde{\gamma}_{pq'}) + \zeta/8 < \mathcal{L}_\omega(\gamma_{pq'}) - 7\zeta/8 < \mathcal{L}_\eta(\gamma_{pq_\eta}^\eta) - 6\zeta/8$$

So the concatenation $\tilde{\gamma}_{pq'} \cdot \gamma_{q'q_\eta}^d$ is a path from p to q_η with length $\mathcal{L}_\eta(\tilde{\gamma}_{pq'} \cdot \gamma_{q'q_\eta}^d) < \mathcal{L}_\eta(\gamma_{pq_\eta}^\eta) - 6\zeta/8 + \zeta/4 = \mathcal{L}_\eta(\gamma_{pq_\eta}^\eta) - \zeta/2$, contradicting the minimality of $\gamma_{pq_\eta}^\eta$. Hence $\mathcal{L}_\omega(\gamma_{pq'}) = d(p, q')$. \square

Proof of Proposition 1.1. Fix a small $\varepsilon > 0$, and choose a tubular neighborhood E of D such that $K = X \setminus E$ is ε -dense with respect to the distance d and $Vol(E, \omega) < \varepsilon^{4n}$. The proof of the Gromov-Hausdorff convergence is completed in two steps:

Claim-1: There exists a $\eta_0 = \eta_0(\varepsilon) > 0$ such that $\forall \eta < \eta_0$, K is ε -dense with respect to d_η . *Proof.* If not, then there exists a sequence $p_\eta \in E$ such that $B_{d_\eta}(p_\eta, \varepsilon) \subset E$. Using volume comparison, diameter bound and the fact that volumes of balls converge, for some uniform $\kappa > 0$ and η small,

$$\kappa\varepsilon^{2n} < Vol_{\omega_\eta}(B_{d_\eta}(p_\eta, \varepsilon)) < Vol_{\omega_\eta}(E) < 2Vol_\omega(E) < 2\varepsilon^{4n}$$

which is a contradiction if ε is small.

Claim-2: There exists a $\eta_0 = \eta_0(\varepsilon) > 0$ such that $\forall \eta < \eta_0$ and for all $p, q \in K$,

$$|d_\eta(p, q) - d(p, q)| < \varepsilon$$

Proof. Let $\tilde{\varepsilon} = d(\partial K, D)/4$, so that in particular $\tilde{\varepsilon} < \varepsilon/4$. We first claim that a neighborhood T of D can be chosen with $Vol(\partial T, \omega)$ arbitrarily small. This can be done because D has real co-dimension two. For a unit polydisc in \mathbb{C}^n with a model edge metric with cone angle $2\pi\beta_j$ along $[z_j = 0]$, such a neighborhood can be constructed explicitly. One can then glue together these local neighborhoods to obtain a neighborhood of D in X with the volume of the boundary arbitrarily small. In particular one can construct a T such that $d(\partial T, K) > 2\tilde{\varepsilon}$ and $Vol(\partial T, \omega) < \delta$ where $\delta = \delta(n, \tilde{\varepsilon}, A, \Lambda)$ is the constant in Lemma 3.2.

Next, by Lemma 3.2, for all $p, q \in K$ there exists $q' \in B_d(q, \tilde{\varepsilon})$ and a minimal d -geodesic $\gamma_{pq'}^d \subset X \setminus T$. Like in the argument for the proof of Lemma 3.2, for η small, $d_\eta(q, q') < 2\tilde{\varepsilon}$.

Then by uniform smooth convergence on $X \setminus T$, there exists $\eta_0 > 0$ such that for $\eta < \eta_0$ and all $p, q \in K$,

$$d_\eta(p, q) < \mathcal{L}_\eta(\gamma_{pq'}^d) + d_\eta(q, q') < \mathcal{L}_\omega(\gamma_{pq'}^d) + 3\tilde{\varepsilon} = d(p, q') + 3\tilde{\varepsilon} < d(p, q) + 4\tilde{\varepsilon} < d(p, q) + \varepsilon$$

On the other hand, recall that $\gamma_{pq'}^d$ is constructed as the limit of η -minimal geodesics $\gamma_{pq_\eta}^\eta \subset X \setminus T$ with $q_\eta \in B_{d_\eta}(q, \tilde{\varepsilon}/2) \subset B_d(q, \tilde{\varepsilon})$, and $q_\eta \rightarrow q'$. So,

$$d(p, q) < d(p, q') + \tilde{\varepsilon} < \mathcal{L}_\eta(\gamma_{pq_\eta}^\eta) + 2\tilde{\varepsilon} = d_\eta(p, q_\eta) + 2\tilde{\varepsilon} < d_\eta(p, q) + 5\tilde{\varepsilon}/2 < d_\eta(p, q) + \varepsilon$$

This finishes the proof of Claim-2.

We can now complete the proof of the theorem. For small $\eta > 0$,

$$\begin{aligned} & d_{GH}((X, d_\eta), (X, d)) \\ & \leq d_{GH}((X, d_\eta), (K, d_\eta)) + d_{GH}((K, d_\eta), (K, d)) + d_{GH}((K, d), (X, d)) \\ & < 3\varepsilon, \end{aligned}$$

where we use Claim 1 to bound the first term, Claim 2 to bound the second term, while the last term is bounded by ε from the choice of K . Now, letting ε go to zero, we see that (X, d_η) converges in Gromov-Hausdorff topology to (X, d) . \square

4. Estimate on volume density and proof of geodesic convexity

Theorem 1.1 follows from a theorem of Colding-Naber [11]. In the previous section, we proved that (X, d) is the Gromov-Hausdorff limit of smooth Riemannian metrics. The crucial point in proving geodesic convexity is that the regular set in the sense of Cheeger-Colding [4] coincides with $X \setminus D$, and hence is open. To prove this, we need to show that the volume density of balls in (X, d) centered on the divisor is strictly less than the Euclidean volume density. We do this by reducing to the case of a smooth divisor (i.e when $N = 1$), and using known regularity results in this situation [19, 8]. The volume density function of an m -dimensional Riemannian manifold (M^m, g) at a $p \in M$ is defined as

$$V_g(p, r) := \frac{\text{Vol}_g(B_g(p, r))}{r^m}$$

We first observe the following elementary fact.

Lemma 4.1. *Let ω_β denote the model edge metric on \mathbb{C}^n with cone angle $2\pi\beta$ along $[z_1 = 0]$ i.e*

$$\omega_\beta = \frac{\sqrt{-1}}{2} \left(\beta^2 |z_1|^{-2(1-\beta)} dz_1 \wedge d\bar{z}_1 + \sum_{j=2}^n dz_j \wedge d\bar{z}_j \right)$$

Then for any $r > 0$,

$$V_{\omega_\beta}(0, r) = \alpha(n)\beta$$

where $\alpha(n) = \pi^n/n!$ is the volume of the unit Euclidean ball in \mathbb{C}^n .

Proof. The ω_β -minimal geodesic connecting the origin to any $(z_1, z') := (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ is given by $\gamma(t) = (t^{1/\beta} z_1, t z_2, \dots, t z_n)$, and it is easily seen that $\mathcal{L}_{\omega_\beta}(\gamma) = |z_1|^{2\beta} + |z'|^2$. So

$B_{\omega_\beta}(0, r) = \{z \in \mathbb{C}^n \mid |z_1|^{2\beta} + |z'|^2 < r\}$. But then, using polar coordinates $z_j = \rho_j e^{i\theta_j}$, and the change of variables $u = \rho^{2\beta}$ in the third line,

$$\begin{aligned}
r^{2n} V_{\omega_\beta}(0, r) &= \int_{B_{\omega_\beta}(0, r)} \frac{\omega_\beta^n}{n!} \\
&= \beta^2 (2\pi)^n \int_{\rho_1^{2\beta} + \dots + \rho_n^2 < r} \frac{(\rho_1 d\rho_1)(\rho_2 d\rho_2) \cdots (\rho_n d\rho_n)}{\rho_1^{2(1-\beta)}} \\
&= \beta (2\pi)^n \int_{u^2 + \dots + \rho_n^2 < r} (udu)(\rho_2 d\rho_2) \cdots (\rho_n d\rho_n) \\
&= \beta \int_0^{2\pi} \cdots \int_0^{2\pi} \int_{u^2 + \dots + \rho_n^2 < r} (udud\theta_1)(\rho_2 d\rho_2 d\theta_2) \cdots (\rho_n d\rho_n d\theta_n) \\
&= \beta \alpha(n) r^{2n}
\end{aligned}$$

□

Lemma 4.2. *Suppose D is a smooth divisor with defining section s , h a smooth Hermitian metric on $[D]$, and $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}$, a Kähler current solving*

$$\tilde{\omega}^n = \frac{\tilde{\Omega}}{|s|_h^{2(1-\beta)}}$$

for some smooth volume form $\tilde{\Omega}$ with $\beta \in (0, 1)$. Then for any $p \in D$,

$$\lim_{r \rightarrow 0} V_{\tilde{\omega}}(p, r) = \beta \alpha(n)$$

Proof. By Prop. 26 in [8], for all $\zeta > 0$, there exists an $r_\zeta > 0$ such that in some holomorphic coordinates centered at $p \in D$,

$$(1 - \zeta)\omega_\beta < \tilde{\omega} < (1 + \zeta)\omega_\beta$$

on $B_{\tilde{\omega}}(p, r_\zeta)$. In [8], the metrics under consideration are conical Kähler-Einstein metrics. A key technical point is the observation that the conical re-scalings of $\tilde{\omega}$ defined by $\tilde{\omega}_\epsilon = \epsilon^{-2} T_\epsilon^* \tilde{\omega}$, where $T_\epsilon(z_1, \dots, z_n) = (\epsilon^{1/\beta} z_1, \epsilon z_2, \dots, \epsilon z_n)$, converge to a metric cone on \mathbb{C}^n . In the present context, by Proposition 1.1 one can approximate $\tilde{\omega}$ by smooth metrics with uniform Ricci lower bound. Then the convergence of the re-scalings to a metric cone is a consequence of general results of Cheeger-Colding [4]. Now it is easy to see that

$$\left(\frac{1-\zeta}{1+\zeta}\right)^n V_\beta(0, \frac{r}{\sqrt{1+\zeta}}) < V_{\tilde{\omega}}(p, r) < \left(\frac{1+\zeta}{1-\zeta}\right)^n V_\beta(0, \frac{r}{\sqrt{1-\zeta}}); \quad \forall r < r_\zeta$$

Lemma 4.1 followed by letting $\zeta \rightarrow 0$ completes the proof. □

Proposition 4.1. *There exists an $\zeta > 0$ and $r(\zeta) > 0$, such that for any $r < r(\zeta)$, and any $p \in D$,*

$$V_d(p, r) := \frac{\text{Vol}(B_d(p, r))}{r^{2n}} < (1 - \zeta)\alpha(n)$$

where $\alpha(n) = \pi^n/n!$ is the volume of the unit Euclidean ball in \mathbb{C}^n .

Proof. The Proposition is proved by smoothening out all but one divisor, and using Lemma 4.2. Without loss of generality, let $p \in D_1$, and consider the equation

$$\begin{cases} \omega_\epsilon^n = \frac{e^{-\lambda\psi_\epsilon - f_\epsilon + c_\epsilon \Omega}}{|s_1|_{h_1}^{2(1-\beta_1)}} \\ \omega_\epsilon = \hat{\omega} + \sqrt{-1}\partial\bar{\partial}\varphi_\epsilon > 0 \end{cases}$$

where ψ_ϵ is the sequence approximating φ from section 2, $f_\epsilon = \log \left(\prod_{j=2}^N (|s_j|_{h_j}^2 + \epsilon)^{(1-\beta_j)} \right)$ and c_ϵ is a constant such that the integrals match up. By Prop. 1.1 there exists a sequence $\omega_{\epsilon,\eta}$ of smooth Kähler metrics and constants A and L such that

$$\begin{aligned} Ric(\omega_{\epsilon,\eta}) &> -A\omega_{\epsilon,\eta} ; \quad diam(X, \omega_{\epsilon,\eta}) < L \\ \omega_{\epsilon,\eta} &\xrightarrow{C_{loc}^\infty(X \setminus D_1)} \omega_\epsilon \\ (X, \omega_{\epsilon,\eta}) &\xrightarrow{d_{GH}} (X, \omega_\epsilon) \end{aligned}$$

By the Bishop-Gromov comparison theorem for the metrics $\omega_{\epsilon,\eta}$ and Colding's convergence theorem [10], for $r' < r$

$$\frac{V_{\omega_\epsilon}(p, r)}{V_{-A}(\tilde{p}, r)} \leq \frac{V_{\omega_\epsilon}(p, r')}{V_{-A}(\tilde{p}, r')}$$

where $V_{-A}(\tilde{p}, r)$ is the volume ratio for the space form of constant sectional curvature $-A/(2n-1)$. Taking $r' \rightarrow 0$, by Lemma 4.2

$$\begin{aligned} V_{\omega_\epsilon}(p, r) &\leq \beta_1 V_{-A}(\tilde{p}, r) \\ &\leq \frac{1 + \beta_1}{2} \alpha(n) \end{aligned}$$

if $r < \bar{r} = \bar{r}(A)$. Moreover, since the Ricci lower bounds for $\omega_{\epsilon,\eta}$ are uniform, by an elementary diagonalization argument, $\omega_\epsilon \xrightarrow{d_{GH}} \omega$ as $\epsilon \rightarrow 0$. Then once again by Colding's theorem on volume convergence, for $r < \bar{r}$,

$$V_d(p, r) < \frac{1 + \beta_1}{2} \alpha(n)$$

This proves the proposition with $\zeta = \frac{1}{2} \max((1 - \beta_1), \dots, (1 - \beta_N))$ and $r(\zeta) = \bar{r}$. \square

Since (X, d) is the Gromov-Hausdorff limit of (X, ω_η) , one can talk about the regular set, in the sense of Cheeger-Colding. It is defined as

$$\mathcal{R} = \{p \in X \mid (X, r_j^{-2}d, p) \xrightarrow{d_{GH}} (\mathbb{C}^n, d_{euc}, 0) \text{ for any sequence } r_j \rightarrow 0\}$$

Lemma 4.3. \mathcal{R} is open and dense in (X, d) .

Proof. By smooth convergence away from D , it is clear that $X \setminus D \subset \mathcal{R}$. On the other hand, suppose $p \in \mathcal{R}$, then by Colding's volume convergence, the volume density $V_d(p, r)$ can be made arbitrarily close to the $\alpha(n)$ for a small enough $r > 0$. But then p cannot belong to D since this would contradict with Proposition 4.1. Hence $\mathcal{R} = X \setminus D$, and is consequently open. The denseness follows from the fact that $X \setminus D$ has full measure. \square

Proof of Theorem 1.1. We follow the line of argument in [11]. By Colding and Naber's result on the Hölder continuity of the tangent cones of limiting spaces of sequences with a Ricci lower bound [11, Cor. 1.5], the set of regular points in the interior of a limiting geodesic is closed. On the other hand, by the above lemma, this set is also open. Therefore, as soon as one interior point lies in $X \setminus D$, all must, and the theorem is proved. \square

Proof of Corollary 1.1. For every $\eta > 0$, there exists a unit speed η -minimal geodesic $\gamma^\eta : [0, l_\eta] \rightarrow X$ connecting p and q with $l_\eta \rightarrow l$. By the Ascoli-Arzelà theorem for Gromov-Hausdorff limits, there exists a continuous limiting geodesic $\gamma : [0, l] \rightarrow X$ connecting p and q . By Theorem 1.1, γ stays away from D . For any $\gamma(t_0)$ with, there exists a small ball $B_d(\gamma(t_0), \epsilon) \subset X \setminus D$. By the argument of Claim-1 in the proof of Lemma 3.2 $B_{d_\eta}(\gamma(t_0), \epsilon/2) \subset B_d(\gamma(t_0), \epsilon)$ for η small enough. By convergence of geodesics and the fact that the geodesics are of unit speed, there exists a δ such that, for η small enough $\gamma^\eta(t) \in B_d(\gamma(t_0), \epsilon)$ for all $|t - t_0| < \delta$. By smooth convergence of the metrics on $X \setminus D$, it is easily seen that $\gamma|_{(t_0-\delta, t_0+\delta)}$ must be smooth, and hence all of γ must be smooth. \square

5. Comparison theorems for conical metrics along simple normal crossing divisors

For this section we fix D to be an effective simple normal crossing divisor given by (1.1), and ω to be a conical Kähler metric along D inducing the metric d on X . The aim of this section is to present extensions of some classical comparison theorems to this singular setting. The crucial point is that the cut locus has measure zero. This is already proved in [18, 11]. For the convenience of the reader, we offer an elementary proof in the conical case exploiting smooth convergence away from the divisor.

Definition 5.1. *We say that*

$$\text{Ric}(\omega) > -A\omega$$

if there exists a smooth positive closed $(1,1)$ form χ such that

$$\text{Ric}(\omega) + A\omega = \chi + [D]$$

For a point $p \in X \setminus D$, let

$$\mathcal{E}_p = \{v \in T_p X \mid \exists \text{ geodesic } \gamma : [0, 1] \rightarrow X \setminus D \text{ with } \gamma(0) = p, \gamma'(0) = v\}$$

The exponential map is well defined and smooth on \mathcal{E} . The following lemma follows directly from Theorem 1.1.

Lemma 5.1. *$\exp_p : \mathcal{E}_p \rightarrow X \setminus D$ is surjective.*

We define the cut locus and conjugate locus in the usual way.

Definition 5.2. (1) *For a $p \in X \setminus D$, the cut locus is defined by*

$$\mathcal{C}_p = \{x \in X \mid \forall z \in X \setminus \{x\}, d(p, x) + d(x, z) > d(p, z)\}$$

(2) *The conjugate locus is defined by*

$$\text{Conj}(p) = \{x \in X \setminus D \mid \exists v \in \exp_p^{-1}(x) \text{ such that } d\exp_p \text{ is degenerate at } v\}$$

Furthermore, $x = \gamma(t_0)$ is said to be conjugate to p along a unit speed geodesic $\gamma : [0, l] \rightarrow X \setminus D$ if \exp_p is singular at $v = t_0 \gamma'(0)$.

The following useful characterization of the cut locus from standard Riemannian geometry [3] also extends to this setting.

Lemma 5.2. *Let $\gamma : [0, l] \rightarrow X \setminus D$ be a smooth unit-speed geodesic emanating from p . Then $x = \gamma(t_0) \in \mathcal{C}_p$ if and only if one of the following holds at $t = t_0$ and neither holds for any smaller value of t :*

- (1) *x is conjugate to p along γ .*
- (2) *There exists a unit speed minimal limiting geodesic $\sigma \neq \gamma$ connecting p and x .*

Proof. Suppose $\gamma(t_0) \in \mathcal{C}_p$, $\epsilon_j \rightarrow 0$ and $\sigma_j : [0, l_j] \rightarrow X \setminus D$ be a unit speed smooth limiting minimal geodesic connecting p to $x_j = \gamma(t_0 + \epsilon_j)$. By continuity of the distance function, $l_j \rightarrow t_0$. By the same argument as in the proof of Cor. 1.1, one can show that there exists a $\sigma : [0, t_0] \rightarrow X \setminus D$ connecting p and x such that $\sigma_j \rightarrow \sigma$ smoothly. If $\sigma \neq \gamma$, criteria (2) is satisfied. If not, then arbitrarily small neighborhoods of $t_0\gamma'(0)$ in \mathcal{E}_p have two distinct vectors, namely $l_j\sigma'_j(0)$ and $(t_0 + \epsilon_j)\gamma'(0)$, mapped to the same point x_j under the exponential map. By the inverse function theorem, $t_0\gamma'(0)$ is a singular point of \exp_p or equivalently x is a conjugate point along γ . \square

As an immediate corollary we have

Corollary 5.1. \mathcal{C}_p has measure zero with respect to ω .

Proof. From the previous Lemma $\mathcal{C}_p \subset \{\text{singular values of } \exp_p\} \cup \{r \text{ is not differentiable}\}$. The first one has measure zero by Sard's theorem, while the second one has measure zero because r is Lipschitz. \square

We now present some classical comparison theorems. We also recall the proofs to emphasize that geodesic convexity, even of the slightly weaker kind proved in the present article, is all that is needed for the extensions to the conical setting.

Theorem 5.1 (Laplacian comparison). *Suppose $\text{Ric}(\omega) > (2n - 1)\lambda\omega$ for some $\lambda \in \mathbb{R}$, and \tilde{X} is the $2n$ -dimensional space form with constant sectional curvature λ . Let $r(x)$ and $\tilde{r}(\tilde{x})$ be distance functions to some fixed points in X and \tilde{X} respectively. Then for any $x \in X \setminus D$ where r is smooth, and any $\tilde{x} \in \tilde{X}$ where \tilde{r} is smooth with $r(x) = \tilde{r}(\tilde{x})$,*

$$\Delta r(x) \leq \tilde{\Delta} \tilde{r}(\tilde{x})$$

Proof. By Bochner formula,

$$\begin{aligned} 0 &= |\nabla^2 r|^2 + \frac{\partial(\Delta r)}{\partial r} + \text{Ric}(\nabla r, \nabla r) \\ &\geq (\Delta r)^2 + \frac{\partial(\Delta r)}{\partial r} + (2n - 1)\lambda \end{aligned}$$

Note that equality holds in the case of \tilde{X} . So, if $\gamma \subset X \setminus D$ and $\tilde{\gamma}$ are unit speed minimal geodesics joining the reference points to x and \tilde{x} respectively, then $u(t) = \Delta r(\gamma(t)) - \tilde{\Delta} \tilde{r}(\tilde{\gamma}(t))$ satisfies the differential inequality

$$\dot{u} + gu \leq 0$$

where $g = \Delta r(\gamma(t)) + \tilde{\Delta} \tilde{r}(\tilde{\gamma}(t))$. Moreover

$$\lim_{t \rightarrow 0} |\Delta r(\gamma(t)) - \left(\frac{2n-1}{t}\right)| = \lim_{t \rightarrow 0} |\tilde{\Delta} \tilde{r}(\tilde{\gamma}(t)) - \left(\frac{2n-1}{t}\right)| = 0$$

i.e $u(0) = 0$. By the method of integrating factors for first order ODEs, it is easily seen that $u(t) \leq 0 \forall t$. \square

Theorem 5.2 (Myer's theorem). *With D as above, suppose ω is a conical Kähler metric along D satisfying $\text{Ric}(\omega) > (2n - 1)\lambda\omega$ for some $\lambda > 0$. Then*

$$\text{diam}(X, d) < \frac{\pi}{\sqrt{\lambda}}$$

Proof. By explicit calculation, if $\lambda > 0$, and \tilde{X} is the space form with sectional curvature λ , then along a unit speed minimal geodesic $\tilde{\gamma}$,

$$\tilde{\Delta}\tilde{r}(\tilde{\gamma}(t)) = (2n-1)\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)}$$

Fix a point $p \in X \setminus D$. For any other point $x \in X \setminus D$, if γ is the minimal unit speed geodesic joining them, then

$$\Delta r(\gamma(t)) \leq (2n-1)\sqrt{\lambda}\frac{\cos(\sqrt{\lambda}t)}{\sin(\sqrt{\lambda}t)}$$

Since right hand side goes to $-\infty$ as $t \rightarrow \pi/\sqrt{\lambda}$, t , and hence the length of γ , can be at most $\pi/\sqrt{\lambda}$. \square

Next, the exponential map is a diffeomorphism from an open subset of \mathcal{E}_p onto $X \setminus (D \cup C_p)$. Moreover, since $C_p \cup D$ has measure zero, standard arguments as in [25] can be used to prove the Bishop-Gromov volume comparison.

Theorem 5.3 (Bishop-Gromov volume comparison). *If $\text{Ric}(\omega) > (2n-1)\lambda\omega$ for some $\lambda \in \mathbb{R}$ and \tilde{X} is the $2n$ -dimensional space form with constant sectional curvature λ . Then*

- (1) *If $K \subset X \setminus D$ is any star convex set centered at x , then for $0 < r_1 < r_2 (< \pi/\sqrt{\lambda}$ if $\lambda > 0$),*

$$\frac{\text{Vol}(B_d(x, r_2) \cap K) - \text{Vol}(B_d(x, r_1) \cap K)}{\tilde{V}(r_2) - \tilde{V}(r_1)} \leq \frac{\text{Vol}(\partial B_d(x, r_1) \cap K)}{\text{Vol}(\partial \tilde{B}(r_1))}$$

where $\tilde{B}(r)$ is a ball of radius r in \tilde{X} and $\tilde{V}(r) = \text{Vol}(\tilde{B}(r))$.

- (2) *For all $x \in X$, the volume ratio*

$$\frac{\text{Vol}(B_d(x, r))}{\tilde{V}(r)}$$

is non-increasing in r .

Remark 5.1. *As a corollary to Theorem 5.3 above, Lemma 3.1 generalizes to Kähler currents satisfying equation (1.5), and in particular to conical Kähler-Einstein metrics. This was very useful in [14] to study the degeneration of conical Kähler-Einstein metrics on toric manifolds.*

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* DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854
E-mail address: veddatar@math.rutgers.edu